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Nonlinear unified equations for water waves propagating over uneven bottoms in the nearshore region*

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Abstract Considering the continuous characteristics for water waves propagating over complex topography in the nearshore region, the unified nonlinear equations, based on the hypothesis for a typical uneven bottom, are presented by employing the Hamiltonian variational principle for water waves. It is verified that the equations include the following special cases: the extension of Airy's nonlinear shallow-water equations, the generalized mild-slope equation, the dispersion relation for the second-order Stokes waves and the higher order Boussinesq-type equations.

Keywords: unified equations, Hamiltonian variational principle for water waves, extended mild-slope equation, higher order Boussinesq-type equations.

For a long time people have tried to establish unified equations for water wave propagation from deep to shallow water, including the mild-slope equation[1], higher order Boussinesq-type equations[2~5], and the Green-Naghdi theory[6], in order to break through the combined wave theories consisting mainly of Stokes deep-water wave theories and shallow-water wave theories. This is most important for realizing the rational distribution and the optimal design of offshore and coastal engineering projects.

Because of the complexity of wave propagation in the nearshore region, the unified theories have the following limitations. The mild-slope equation can only be used for describing linear monochromatic waves, but not for finite-amplitude waves. On account of the drawback of weak nonlinearity and weak dispersion, the higher-order Boussinesq-type equations cannot provide accurate numerical results when the water depth is approximate or equal to a wave length. The Green-Naghdi theory can be regarded as a new type of water wave theory with applicable potential, but it can hardly be widely accepted and applied to the practical engineering due to the complexity of its mathematical formulation. In view of these problems, this paper provides the unified equations which include the currently used main wave equation theories, according to the Hamiltonian variational principle for water waves.

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1 Hamiltonian formulation of water waves

Consider a potential flow of an incompressible inviscid fluid of constant density ρ with gravity g. Its surface elevation $\zeta(x, y, t)$ is assumed to be a single-valued function of the horizontal coordinates x and y. And as $x^2 + y^2 \to 0$, ζ together with all its derivatives and the velocity field $\nabla\Phi$ tends to zero. The total energy of the wave motion is given as the sum of the potential energy and the kinetic energy:

$$H = \frac{1}{2}\rho\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_0 \mathrm{d}x\mathrm{d}y, \tag{1}$$

where

$$H_0 = \frac{1}{2}g\zeta^2 + \frac{1}{2}\int_{-h}^{\zeta}\mathrm{d}z\left[(\nabla\Phi)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2\right] \tag{2}$$

is the Hamiltonian density, in which $h(x, y)$ denotes the water depth and $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$.

The surface elevation $\zeta(x, y, t)$ and the velocity potential at the free surface $\phi(x, y, t) = \Phi(x, y, \zeta(x, y, t), t)$ are canonical variables of Hamiltonian functional $H(\zeta, \phi)$. Taking the variational derivatives of H, the canonical evolution equations

$$\rho\frac{\partial\zeta}{\partial t} = \frac{\partial H}{\partial\phi}, \tag{3a}$$

$$\rho\frac{\partial\phi}{\partial t} = -\frac{\partial H}{\partial\zeta} \tag{3b}$$

are respectively equivalent to the kinematic and dynamic boundary conditions at the free surface. This is the basic content of the Hamiltonian variational principle. Using the Lagrangian density

$$L_0 = \zeta\frac{\partial\phi}{\partial t} + H_0, \tag{4}$$

we have the Euler-Lagrange equations which are identical to the canonical equations (3a) and (3b)

$$\frac{\partial}{\partial t}\left(\frac{\partial L_0}{\partial\phi_t}\right) + \nabla\cdot\left(\frac{\partial L_0}{\partial\nabla\phi}\right) - \frac{\partial L_0}{\partial\phi} = 0, \tag{5a}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L_0}{\partial\zeta_t}\right) + \nabla\cdot\left(\frac{\partial L_0}{\partial\nabla\zeta}\right) - \frac{\partial L_0}{\partial\zeta} = 0. \tag{5b}$$

2 Derivation of the nonlinear unified equations

The mild-slope equation is based on the mild-slope assumption, but the uneven bottoms involve the abrupt-slope topography as well, such as rippled beds, sand bars, and offshore reefs. Thus we now adopt Liu and Dingemans's choice[7] for the depth variations consisting of a slowly varying component $h_0(\boldsymbol{x})$ and a fast varying component $h_1(\boldsymbol{x})$ $(\boldsymbol{x} \equiv (x, y))$:

$$h = h_0(\beta\boldsymbol{x}) + \gamma^2 h_1(\boldsymbol{x}), \tag{6}$$

$$O(\beta) = O\left(\frac{\nabla h_0}{k_0 h_0}\right) = O\left(\frac{\lambda}{\Lambda}\right) \ll 1, \; O(kh_1) \approx O(\gamma^2), \; O\left(\frac{h_1}{\Lambda}\right) = O(\gamma^3). \tag{7}$$

Parameter Λ, the horizontal length scale for $h_0(\boldsymbol{x})$, is much longer than a typical wave length of the surface waves. The variation of h_0 within a wave length is small. The horizontal length scales of h_1 are in the same order of magnitude as the wave length and the size of h_1 is, however, small. Symbols β and γ denote the small modulation parameters and are assumed to be of the same order of magnitude, i. e. $O(\beta) = O(\gamma)$, and $\lambda = 2\pi/k$ is a wave length of the surface wave.

With respect to the structure of the solution for linear waves, the velocity potential can be approximately expressed as[8]

$$\Phi(\boldsymbol{x}, z, t) = \frac{\cosh k(h + z)}{\cosh k(h + \zeta)} \phi(\boldsymbol{x}, t) \equiv f\phi. \tag{8}$$

The characteristic value k is the positive solution of the equation

$$\omega^2 = gk \tanh k(h_0 + \zeta), \tag{9}$$

and clearly $\Phi(\boldsymbol{x}, \zeta, t) = \boldsymbol{\phi}(x, y, t)$. When $\zeta = 0$, f is just the solution of linear waves; and Eq. (9) is changed into the usual linear dispersion relation. We expand $f(z, h)$ into a Taylor's series about $h = h_0$ and obtain

$$f(z, h) = f_0(z, h_0) + \gamma^2 h_1 f_1(z, h_0), \tag{10}$$

$$f_0(z, h_0) = \frac{\cosh k(z + h_0)}{\cosh k(h_0 + \zeta_0)}, \quad f_1(z, h_0) = \frac{k \sinh k(z - \zeta_0)}{\cosh^2 k(h_0 + \zeta_0)}. \tag{11}$$

From Eq. (2), we have

$$\begin{aligned} H_0 = {} & \frac{1}{2} g\zeta^2 + \frac{1}{2}(\nabla\phi \cdot \nabla\phi)\left(\frac{1}{k}\alpha_1 - \gamma^2 \alpha_2 h_1\right) \\ & + \frac{1}{2}\phi^2[(\alpha_3 k + \gamma^2 \alpha_4 h_1 k^2)(\nabla\zeta \cdot \nabla\zeta) + \alpha_5 k(\nabla h_0 \cdot \nabla h_0) \\ & + \alpha_6 k \nabla\zeta \cdot (\nabla h_0 + \gamma^2 \nabla h_1) + \alpha_7 k + \gamma^2 \alpha_2 h_1 k^2] - \phi\nabla\phi \cdot \Big\{\alpha_1[\sigma + \gamma^2 k h_1 (1 - \sigma^2)]\nabla\zeta \\ & + \frac{1}{2}\alpha_2[\nabla h_0 + \gamma^2(\nabla h_1 - 2\sigma k h_1 \nabla\zeta)]\Big\} + \frac{1}{2}\gamma^2 h_1 \Big\{\alpha_8(\nabla\phi \cdot \nabla\phi) \\ & + \alpha_9 k^2 \phi^2(\nabla\zeta \cdot \nabla\zeta + 2\nabla\zeta \cdot \nabla h_0) - 2\phi\nabla\phi \cdot [\alpha_{10} k(\nabla\zeta + \nabla h_0) + \alpha_{11} h_1 k^2 \nabla\zeta]\Big\}, \end{aligned} \tag{12}$$

in which $\sigma = \tanh k(h_0 + \zeta)$. And the detailed expressions for the dimensionless parameters $\alpha_i (i = 1, 2, \cdots 11)$ are given as follows:

$$\alpha_1 = \frac{1}{2}[\sigma + q(1 - \sigma^2)], \; \alpha_2 = q\sigma(1 - \sigma^2), \; \alpha_3 = \frac{1}{2}\sigma^2[\sigma + q(1 - \sigma^2)],$$

$$\alpha_4 = \sigma(1 - \sigma^2)(q + \sigma - 2q\sigma^2), \; \alpha_5 = \frac{1}{2}(-q + \sigma + 2q\sigma^2 - \sigma^3 - q\sigma^4), \; \alpha_6 = q\sigma^2(1 - \sigma^2),$$

$$\alpha_7 = \frac{1}{2}[\sigma - q(1 - \sigma^2)],\ \alpha_8 = 1 - \sigma^2,\ \alpha_9 = \sigma^2(1 - \sigma^2),$$

$$\alpha_{10} = \sigma(1 - \sigma^2),\ \alpha_{11} = (1 - \sigma^2)^2,$$

where $q = k(\zeta + h_0)$.

Combining Eqs. (3a), (3b) and (12), the nonlinear unified equations for water waves propagating over uneven bottoms can be obtained:

$$\frac{\partial \zeta}{\partial t} = \frac{\partial H_0}{\partial \phi} - \nabla \cdot \left(\frac{\partial H_0}{\partial \nabla \phi}\right), \tag{13a}$$

$$\frac{\partial \phi}{\partial t} = \frac{\partial H_0}{\partial \zeta} + \nabla \cdot \left(\frac{\partial H_0}{\partial \nabla \zeta}\right). \tag{13b}$$

Eqs. (13a) and (13b) are implicit expressions.

3 Special cases

Now let us analyze the explicit expressions for Eqs. (13a) and (13b) under some special conditions, and relate them to a number of currently used equation theories for water waves.

3.1 Generalized Airy's nonlinear shallow-water equations

For finite-amplitude and very long waves, i.e. $\varepsilon = \frac{a}{h_0} = O(1)$, $\mu = (kh_0)^2 \ll 1$, where a is wave amplitude, from Eq. (12) we have

$$H_0 = \frac{1}{2}g\zeta^2 + \frac{1}{2}(h_0 + \zeta + r^2 h_1)(\nabla\phi \cdot \nabla\phi) - \gamma^2 k^2 h_1[h_1 + 2(h_0 + \zeta)]\phi\nabla\phi \cdot \nabla\zeta. \tag{14}$$

Let $U = \nabla\phi$ and combine Eqs. (13a), (13b) and (14). Then the extended Airy's nonlinear shallow-water equations can be given as

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot [(h_0 + \zeta + \gamma^2 h_1)U] - \gamma^2\phi\nabla \cdot \{h_1 k^2[2(h_0 + \zeta) + h_1]\nabla\zeta\} = 0, \tag{15a}$$

$$\frac{\partial U}{\partial t} + U \cdot \nabla U + g\nabla\zeta + \gamma^2\{\nabla\nabla \cdot [h_1 k^2\phi U(2h_0 + h_1)] + 2[\nabla \cdot (h_1 k^2\phi U)\nabla\zeta + \zeta\nabla\nabla \cdot (h_1 k^2 \phi U)]\} = 0. \tag{15b}$$

3.2 Generalized mild-slope equation

For small-amplitude waves, from Eq. (12) we have

$$H_0 = \frac{1}{2}g\zeta^2 + \frac{1}{2}(\nabla\phi \cdot \nabla\phi)\left[\frac{1}{k}\alpha'_1 + \gamma^2(\alpha'_8 - \alpha'_2)h_1\right] + \frac{1}{2}\phi^2[\alpha'_7 k + \alpha'_5 k(\nabla h_0 \cdot \nabla h_0) + \gamma^2\alpha'_2 h_1 k^2] - \phi\nabla\phi \cdot \left[\frac{1}{2}\alpha'_2\nabla h_0 + \gamma^2\left(\frac{1}{2}\alpha'_2\nabla h_1 + \alpha'_{10}kh_1\nabla h_0\right)\right]. \tag{16}$$

Making $\zeta = 0$ in the α_i terms leads to the corresponding terms α'_i. From Eqs. (13a), (13b), and

(16), we get

$$\frac{\partial \zeta}{\partial t} = \phi\left\{\alpha'_7 k + \alpha'_5 k(\nabla h_0 \cdot \nabla h_0) + \gamma^2 \alpha'_2 h_1 k^2 + \nabla \cdot \left[\frac{1}{2}\alpha'_2 \nabla h_0 + \gamma^2\left(\frac{1}{2}\alpha'_2 \nabla h_1 + k h_1 \alpha'_{10} \nabla h_0\right)\right]\right\} - \nabla \cdot \left\{\left[\frac{1}{k}\alpha'_1 + \gamma^2 h_1(\alpha'_8 - \alpha'_2)\right]\nabla \phi\right\}, \tag{17a}$$

$$\frac{\partial \phi}{\partial t} = - g\zeta. \tag{17b}$$

Eliminating ζ from (17a) and (17b) and ignoring the resulting higher-order terms of $O(\gamma^2 \beta^2)$, we have generalized time-dependent mild-slope equation

$$\frac{\partial^2 \phi}{\partial t} + (\omega^2 - k^2 CC_g)\phi - \nabla \cdot (CC_g \nabla \phi) + g[F_1 \phi - \nabla \cdot (F_2 \nabla \phi)] = 0, \tag{18}$$

where

$$F_1 = \alpha'_5 k(\nabla h_0 \cdot \nabla h_0) + \frac{1}{2}\nabla \cdot (\alpha'_2 \nabla h_0 + \gamma^2 \alpha'_2 \nabla h_1) + \gamma^2[\alpha'_2 h_1 k^2 + \nabla h_0 \cdot \nabla(\alpha'_{10} k h_1)], \tag{19}$$

$$F_2 = \gamma^2 h_1(\alpha'_8 - \alpha'_2). \tag{20}$$

Equations similar to Eq. (18) were obtained by Kirby[9] and Dingemans[10] on different assumptions for fast varying topography, and by Huang et al.[11] who considered the effects of both uneven bottoms and ambient currents.

3.3 Stokes waves theory

For deep-water waves: $kh \rightarrow \infty$ such that $\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{2}$, and the remaining α_i terms are all zero. Thus from Eq. (12) we obtain

$$H_0 = \frac{1}{2} g\zeta^2 + \frac{1}{4k}(\nabla \phi - k\phi \nabla \zeta)^2 + \frac{1}{4} k\phi^2. \tag{21}$$

The canonical equations can be obtained with Eqs. (13a) and (13b):

$$\frac{\partial \zeta}{\partial t} + \frac{1}{2k}(\nabla^2 \phi - k\phi \nabla^2 \zeta)^2 + \frac{1}{2}k(1 + \nabla \zeta \cdot \nabla \zeta)\phi = 0, \tag{22a}$$

$$\frac{\partial \phi}{\partial t} + g\zeta + \frac{1}{2}\nabla \cdot [\phi(\nabla \phi - k\phi \nabla \zeta)] = 0. \tag{22b}$$

From Eq. (21) we can infer the famous Stokes second-order dispersion relation[8]

$$\omega = \sqrt{gk}\left(1 + \frac{1}{2}k^2 a^2\right). \tag{23}$$

3.4 Higher order Boussinesq-type equations

There are two important parameters for shallow water: the nonlinearity, $\varepsilon = \frac{\alpha}{h}$, and the dispersion, $\mu = (k\bar{h})$, where $\bar{h}$ is the average water depth. We use dimensionless variables.

$$x' = Rx', \ h'_0 = \frac{h_0}{\bar{h}}, \ h'_1 = \frac{h_1}{\bar{h}},$$

$$\zeta' = \frac{\zeta}{a}, \ \phi' = \frac{k}{\varepsilon\sqrt{gh}}\phi, \ \nabla' = \left(\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}\right). \tag{24}$$

From Eq. (4) we have

$$L_0 = (ga^2)\left(\zeta'\frac{\partial\phi'}{\partial t'} + H'_0\right) \equiv (ga^2)L'_0, \tag{25}$$

in which H'_0 can be written as (omitting the primes):

$$\begin{aligned}
H'_0 = {} & \frac{1}{2}\zeta^2 + \frac{1}{2}(\nabla\phi\cdot\nabla\phi)\left\{\left[h_0\left(1+\varepsilon\frac{\zeta}{h_0}\right) - \frac{2}{3}\mu h_0^3\left(1+\varepsilon\frac{\zeta}{h_0}\right)^2\right]\right. \\
& \left. - \gamma^2 h_1\left[\mu h_0^2\left(1+\varepsilon\frac{\zeta}{h_0}\right) - \frac{4}{3}\mu^2 h_0^4\left(1+\varepsilon\frac{\zeta}{h_0}\right)^2\right]\right\} \\
& + \frac{1}{2}\phi^2\left\{\varepsilon^2(\nabla\zeta\cdot\nabla\zeta)\left[\mu^2 h_0^3\left(1+\varepsilon\frac{\zeta}{h_0}\right) + 2\gamma^2\mu^2 h_1 h_0^2\right]\right. \\
& + \frac{1}{3}\mu^2 h_0^3(\nabla h_0\cdot\nabla h_0)\left(1+\varepsilon\frac{\zeta}{h_0}\right) \\
& \left. + \varepsilon\mu^2 h_0^3\left(1+\varepsilon\frac{\zeta}{h_0}\right)\nabla\zeta\cdot(\nabla h_0 + \gamma^2\nabla h_1)\right\} - \phi\nabla\phi\cdot\{\varepsilon\mu h_0^2(\nabla\zeta) \\
& \left[\left(1+\varepsilon\frac{\zeta}{h_0}\right) - \mu h_0^2\left(1+2\varepsilon\frac{\zeta}{h_0}\right) + \frac{2}{9}\mu^2 h_0^4\right. \\
& \left. + \frac{\gamma^2 h_1}{h_0}\left(1 - \frac{5}{3}\mu h_0^2\left(1+\varepsilon\frac{\zeta}{h_0}\right) + \frac{2}{3}\mu^2 h_0^4\right)\right] \\
& + \left[\left(\frac{1}{2}\mu h_0^2\left(1+\varepsilon\frac{\zeta}{h_0}\right) - \frac{2}{3}\mu^2 h_0^4\left(1+\varepsilon\frac{\zeta}{h_0}\right)^2\right)\nabla h_0\right. \\
& + \gamma^2\left(\left(\frac{1}{2}\mu h_0^2\left(1+\varepsilon\frac{\zeta}{h_0}\right) - \frac{2}{3}\mu^2 h_0^4\left(1+\varepsilon\frac{\zeta}{h_0}\right)^2\right)\nabla h_1 - \varepsilon\mu_1 h_1 h_0(\nabla\zeta)\right. \\
& \left.\left.\left(\mu h_0^2\left(1+\varepsilon\frac{\zeta}{h_0}\right) - \frac{5}{3}\mu^2 h_0^4\right)\right)\right]\} \\
& + \frac{1}{2}\gamma^2 h_1\left\{(\nabla\phi\cdot\nabla\phi)\left[1 - \mu h_0^2\left(1+\varepsilon\frac{\zeta}{h_0}\right) - \frac{1}{3}\mu^2 h_0^4\left(1+\varepsilon\frac{\zeta}{h_0}\right)^2\right]\right. \\
& + \phi^2\left[\varepsilon^2\mu^2 h_0^2(\nabla\zeta\cdot\nabla\zeta) + 2\varepsilon\mu^2 h_0^2(\nabla\zeta\cdot\nabla h_0)\right. \\
& \left. - \frac{8}{3}\varepsilon\mu^3 h_0^4(\nabla\zeta\cdot\nabla h_0)\right] - 2\phi\nabla\phi\cdot\left[(\varepsilon\nabla\zeta + \nabla h_0)\left(\mu h_0 - \frac{4}{3}\mu^2 h_0^3\left(1+\varepsilon\frac{\zeta}{h_0}\right)\right)\right. \\
& \left.\left. + \varepsilon h_1(\nabla\zeta)\left(\mu - 2\mu^2 h_0^2 + \frac{5}{3}\mu^3 h_0^4\right)\right]\right\} + \frac{1}{2}\phi^2\left\{\frac{1}{3}\mu h_0^3\left(1+\varepsilon\frac{\zeta}{h_0}\right)^2\right. \\
& \left. + \gamma^2 h_1\left[\mu h_0^2\left(1+\varepsilon\frac{\zeta}{h_0} - \frac{4}{3}\mu^2 h_0^4\left(1+\varepsilon\frac{\zeta}{h_0}\right)\right)^2\right]\right\}.
\end{aligned} \tag{26}$$

The non-dimensional equations corresponding to Eqs. (5a) and (5b) are formally identical to the latter. Therefore, by omitting the primes, and from Eqs. (5a), (5b), (25), and (26), we have the following equations:

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot \left\{ \nabla \phi \left[h_0 \left(1 + \varepsilon \frac{\zeta}{h_0}\right) - \frac{2}{3} \mu h_0^3 \left(1 + \varepsilon \frac{\zeta}{h_0}\right)^2 - \right. \right.$$

$$\gamma^2 h_1 \left(\mu h_0^2 \left(1 + \varepsilon \frac{\zeta}{h_0}\right) - \frac{4}{3} \mu^2 h_0^4 \left(1 + \varepsilon \frac{\zeta}{h_0}\right)^2 \right) \Big]$$

$$+ \gamma^2 h_1 \nabla \phi \left[1 - \mu h_0^2 \left(1 + \varepsilon \frac{\zeta}{h_0}\right) - \frac{1}{3} \mu^2 h_0^4 \left(1 + \varepsilon \frac{\zeta}{h_0}\right)^2 \right] \Big\}$$

$$- \phi \Big\{ \varepsilon^2 (\nabla \zeta \cdot \nabla \zeta) \left[\mu^2 h_0^3 \left(1 + \varepsilon \frac{\zeta}{h_0}\right) + 2\gamma^2 \mu^2 h_1 h_0^2 \right]$$

$$+ \frac{1}{3} \mu^2 h_0^3 (\nabla h_0 \cdot \nabla h_0) \left(1 + \varepsilon \frac{\zeta}{h_0}\right) + \varepsilon \mu^2 h_0^3 (\nabla \zeta)$$

$$\left(1 + \varepsilon \frac{\zeta}{h_0}\right) \cdot (\nabla h_0 + \gamma^2 \nabla h_1) + \gamma^2 h_1 \Big[\varepsilon^2 \mu^2 h_0^2 (\nabla \zeta \cdot \nabla \zeta)$$

$$+ 2\varepsilon \mu^2 h_0^2 (\nabla \zeta \cdot \nabla h_0) - \frac{8}{3} \varepsilon \mu^3 h_0^4 (\nabla \zeta \cdot \nabla h_0) \Big] + \frac{1}{3} \mu h_0^3$$

$$\left(1 + \varepsilon \frac{\zeta}{h_0}\right)^2 + \gamma^2 h_1 \Big[\mu h_0^2 \left(1 + \varepsilon \frac{\zeta}{h_0}\right)$$

$$- \frac{4}{3} \mu^2 h_0^4 \left(1 + \varepsilon \frac{\zeta}{h_0}\right)^2 \Big] \Big\} - \phi \nabla \cdot \Big\{ \varepsilon \mu h_0^2 (\nabla \zeta) \Big[\left(1 + \varepsilon \frac{\zeta}{h_0}\right) - \mu h_0^2 \left(1 + 2\varepsilon \frac{\zeta}{h_0}\right) + \frac{2}{9} \mu^2 h_0^4$$

$$+ \frac{\gamma^2 h_1}{h_0} \left(1 - \frac{5}{3} \mu h_0^2 \left(1 + \varepsilon \frac{\zeta}{h_0}\right) + \frac{2}{3} \mu^2 h_0^4 \right) \Big] +$$

$$\Big[\left(\frac{1}{2} \mu h_0^2 \left(1 + \varepsilon \frac{\zeta}{h_0}\right) - \frac{2}{3} \mu^2 h_0^4 \left(1 + \varepsilon \frac{\zeta}{h_0}\right)^2 \right) \nabla h_0$$

$$+ \gamma^2 \left(\left(\frac{1}{2} \mu h_0^2 \left(1 + \varepsilon \frac{\zeta}{h_0}\right) - \frac{2}{3} \mu^2 h_0^4 \left(1 + \varepsilon \frac{\zeta}{h_0}\right)^2 \right) \nabla h_1 \right.$$

$$- \varepsilon \mu h_1 h_0 (\nabla \zeta) \left(\mu h_0^2 \left(1 + \varepsilon \frac{\zeta}{h_0}\right) - \frac{5}{3} \mu^2 h_0^4 \right) \Big) \Big] \Big\}$$

$$- \gamma^2 \phi \nabla \Big\{ h_1 \Big[(\varepsilon \nabla \zeta + \nabla h_0) \left(\mu h_0 - \frac{4}{3} \mu^2 h_0^3 \left(1 + \varepsilon \frac{\zeta}{h_0}\right) \right)$$

$$+ \varepsilon h_1 (\nabla \zeta) \left(\mu - 2\mu^2 h_0^2 + \frac{5}{3} \mu^3 h_0^4 \right) \Big] \Big\} = 0, \tag{27a}$$

$$\frac{\partial \phi}{\partial t} + \zeta + (\nabla \phi \cdot \nabla \phi) \left[\frac{1}{2} \varepsilon - \frac{2}{3} \varepsilon \mu h_0^2 - \gamma^2 h_1 \left(\frac{1}{2} \varepsilon \mu h_0 - \frac{4}{3} \varepsilon \mu^2 h_0^3 \right) \right]$$

$$+ \phi^2 \Big[\frac{1}{3} \varepsilon \mu h_0^2 + \frac{1}{6} \varepsilon \mu^2 h_0^2 (\nabla h_0 \cdot \nabla h_0)$$

$$+ \gamma^2 h_1 \left(\frac{1}{2} \varepsilon \mu h_0 - \frac{4}{3} \varepsilon \mu^2 h_0^3 \right) \Big] - \phi \nabla \phi \cdot \Big[(\nabla h_0) \left(\frac{1}{2} \varepsilon \mu h_0 - \frac{4}{3} \varepsilon \mu^2 h_0^3 \right)$$

$$+ \gamma^2(\nabla h_1)\left(\frac{1}{2}\varepsilon\mu h_0 - \frac{4}{3}\varepsilon\mu^2 h_0^3\right)\Big]$$

$$+ \gamma^2 h_1\Big[(\nabla\phi \cdot \nabla\phi)\left(-\frac{1}{2}\varepsilon\mu h_0 - \frac{1}{3}\varepsilon\mu^2 h_0^3\right) + \frac{4}{3}\varepsilon\mu^2 h_0^2 \phi\nabla\phi \cdot \nabla h_0\Big] -$$

$$\nabla \cdot \Big\{\phi^2\Big[\frac{1}{2}\varepsilon\mu^2 h_0^3(\nabla h_0 + \gamma^2 \nabla h_1) + \gamma^2 h_1(\nabla h_0)\left(\varepsilon\mu^2 h_0^2 - \frac{4}{3}\varepsilon\mu^3 h_0^4\right)\Big]$$

$$- \phi\nabla\phi\Big[\varepsilon\mu h_0^2 - \varepsilon\mu^2 h_0^4 + \frac{2}{9}\varepsilon\mu^3 h_0^6$$

$$+ \gamma^2 h_1\left(2\varepsilon\mu h_0 - 4\varepsilon\mu^2 h_0^3 + \frac{7}{3}\varepsilon\mu^3 h_0^5 + h_1\left(\varepsilon\mu - 2\varepsilon\mu^2 h_0^2 + \frac{5}{3}\varepsilon\mu^3 h_0^4\right)\right)\Big]\Big\} = 0. \quad (27b)$$

Eqs. (27a) and (27b) constitute the new Boussinesq-type equations containing higher order terms than others, thus improving the predictability on the propagation characteristics of water waves.

4 Concluding remarks

In view of the urgent requirement of the unified water wave equation for offshore and coastal engineering practice , the nonlinear unified equations (13a) and (13b) for water waves propagating over typical uneven bottoms are derived using the Hamiltonian variational principle for water waves. Though implicitly given, the unified equations include three kinds of wave equations and one kind of dispersion relation.

Owing to the fact that pure waves rarely exist in the nearshore region, it is imperative to bring the wave-current interaction into the unified equation, that is, developing the unified equations for wave-current interactions over uneven bottoms in the complex dynamical nearshore environment. The work is in progress.

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